# Revisiting roots of real numbers

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## Introduction

We have observed that over 90% of our students, both undergraduate and graduate, know little about the existence and multiplicity of real roots of real numbers; for example the fifth root of –2. Most of those who may know the answers are unable to give a logical explanation of the validity of their answers.

Here we give an elementary but rigorous treatment of this topic, which not only makes the students aware of some important properties of real numbers but can also provide a nice example of a rigorous treatment of a mathematical reasoning at an elementary level that even a competent high school student can understand. It is clear that we need to start emphasizing mathematical arguments and proofs in our classes and discourage total dependence on technology. We need to reconsider a better balance between *what* to teach and *how* to teach.

To establish the existence of roots, we need the *intermediate value theorem*, which is fairly intuitive if one understands the definition of continuity (for the sake of completeness, we state it below). We do not need any more advanced topics or differentiability. We also give an alternate proof of the main existence theorem in the Appendix, using some basic properties of real numbers such as the *completeness property* of real numbers, *trichotomy of inequality* of real numbers, and *Archimedean property*. This method is an extension and modification of the method used in [1], in order to prove existence of the square root of 2. However, they do not address the number of square roots or roots of numbers in general.

**Intermediate value theorem**

Let $f(x)$ be a function defined and continuous on the interval $[a, b]$ and let $c$ be a number such that $f(a) < c < f(b)$. Then there exists a number $s$ in $(a, b)$ such that $f(s) = c$.

To establish the multiplicity of roots we do not need anything more advanced than the formula for factoring the difference of two monomials given by

$$x^n - y^n = (x-y)\left(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + y^{n-1}\right)$$
$$= (x-y)\sum_{j=1}^{j=n} x^{n-j}y^{j-1} \qquad (1)$$

A convincing argument for the proof of this elementary result, which plays a crucial and central role here, can be given by simply multiplying the second quantity inside parentheses first by $x$ and then by $-y$, and noticing that for each $j$, $1 < j < n$, the term $xx^{n-j}\, y^{j-1}$ cancels the term $-yx^{n-(j-1)}y^{j-2}$. However, we include a more rigorous proof, using mathematical induction, below.

**Proof by mathematical induction**

It is obvious that for $n = 1$, the formula

$$x^n - y^n = (x-y)\sum_{j=1}^{j=n} x^{n-j}y^{j-1}$$

holds. Suppose that it is true for $n = k$. We wish to show that it is then also true for $n = k + 1$. By subtracting and adding the term $xy^k$, we obtain (by the induction hypothesis)

$$x^{k+1} - y^{k+1} = xx^k - yy^k$$
$$= xx^k - xy^k + xy^k - yy^k$$
$$= x(x^k - y^k) + y^k(x-y)$$
$$= x(x-y)\sum_{j=1}^{j=k} x^{k-j}y^{j-1} + y^k(x-y)$$

Hence,

$$x^{k+1} - y^{k+1} = (x-y)\left[x\sum_{j=1}^{j=k} x^{k-j}y^{j-1} + y^k\right]$$
$$= (x-y)\sum_{j=1}^{j=k} x^{k+1-j}y^{j-1} + y^k$$
$$= (x-y)\sum_{j=1}^{j=k+1} x^{k+1-j}y^{j-1}$$

Therefore, the formula holds for $n = k + 1$ and the proof is complete.

## A couple of simple examples

We would like to point out that many textbooks prove that $\sqrt{2}$ is irrational by assuming that it is rational and then showing that this assumption leads to a contradiction. However, this argument is quite unsatisfactory unless they first show the existence of $\sqrt{2}$. For example, $\sqrt{-1} = i$ is not rational but it is not irrational either. One of the few books that does deal with the existence issue is the one by Robert G. Bartle and Donald R. Sherbert.

**Example 1**

The number 2 has exactly two real square roots.

We recall that finding a square root of 2 is equivalent to finding the real roots of the equation

$$x^2 - 2 = 0 \tag{2}$$

Let $f(x) = x^2 - 2$. Then, we have $f(2) = 2 > 0$ and $f(1) = -1 < 0$. Therefore, by the intermediate value theorem, there exists a number $r_1$ between 1 and 2 satisfying equation (2). We denote this number by $r_1 = \sqrt{2}$. In order to find a second square root of the number 2, we simply note that if $r_1$ is a square root of 2, then so is $r_2 = -r_1 = -\sqrt{2}$. We have now shown that equation (2) has two distinct roots and hence the number 2 has two distinct square roots $r_1$ and $r_2$. But we have not yet shown that it cannot have more than two. In order to do that, let $r_3$ be any root of (2). Then equations $r_1^2 = 2$ and $r_3^2 = 2$ imply that $r_3^2 = r_1^2$; hence $r_3^2 - r_1^2 = 0$. Therefore, $(r_3 - r_1)(r_3 + r_1) = 0$, which implies that either $r_3 = r_1$ or $r_3 = -r_1$. This shows that there are no more than the two solutions $r_1$ and $r_2$, and hence 2 has exactly two distinct real square roots.

**Example 2**

The number –2 has exactly one real cube root.

Recall that a number $x$ is a cube root of –2 if $x^3 = -2$. Thus, finding the real cube roots of –2 is equivalent to finding the real roots of the equation

$$x^3 + 2 = 0 \tag{3}$$

We note that for $f(x) = x^3 + 2$, $f(0) = 2 > 0$ while $f(-2) = -6 < 0$. Consequently, by the intermediate value theorem, there exists a number $s$ between –2 and 0 satisfying equation (3). This shows that the number $s$ is a cube root of the number –2, which we denote as $s = \sqrt[3]{-2}$. Now we show that the cube root of –2 is unique. Let $t$ be any cube root of –2. We will show that $t = s$. It follows that $t < 0$, since having $t \geq 0$ and $t^3 + 2 = 0$ would be impossible. Similarly, $s < 0$. Now, since $s^3 = -2 = t^3$, it follows that $s^3 - t^3 = 0$. Factoring $s^3 - t^3$, we obtain

$$s^3 - t^3 = (s - t)(s^2 + st + t^2) = 0$$

Since $s$ and $t$ are both negative, we must have $s^2 + st + t^2 > 0$. This implies that we must have $s - t = 0$, or $s = t$. This shows that there is no more than one cube root of –2.

## Roots of real numbers in general

For the existence part, first we prove that every positive number $c$ has a real $n$th root, where $n$ is any natural number, $n \geq 2$. We then use simple logic to completely analyse the existence of $n$th roots of *all* real numbers.

We first prove a simple lemma, not only for the sake of rigor but also because it demonstrates another nice application of our simple but useful factoring formula (1).

**Lemma 1**

If $a$ is a real number such that $a > 1$, then $a^n > a$.

*Proof*

In order to show that $a^n - a > 0$, since $1^{j-1} = 1$, we can use our factoring formula to obtain

$$\begin{aligned} a^n - a &= a\left(a^{n-1} - 1\right) \\ &= a\left(a^{n-1} - 1^{n-1}\right) \\ &= a(a-1)\sum_{j=2}^{j=n} a^{n-j} \end{aligned}$$

Since $a$, $a$–1, and $\sum_{j=2}^{j=n} a^{n-j}$ are all positive, our assertion follows.

**Theorem 1**

Every positive real number $c$ has at least one positive $n$th root, where $n$ is a natural number, $n \geq 2$.

*Proof*

We want to show that the equation $x^n - c = 0$ has a positive root.

Let $f(x) = x^n - c$. We note that $f(0) = -c < 0$.

By Lemma 1, $(c + 1)^n > c + 1$.

Hence $f(c + 1) = (c + 1)^n - c > (c + 1) - c = (c - c) + 1 = 1 > 0$.

Therefore, by the intermediate value theorem, there exists a number $r$ between 0 and $c + 1$ such that $r^n - c = 0$.

The following theorem gives a complete characterisation of the $n$th roots of real numbers.

**Theorem 2**

(a) If $c$ is positive and $n$ is even, then $c$ has exactly two $n$th roots.

(b) If $c$ is negative and $n$ is even, then $c$ has no real $n$th roots.

(c) If $c$ is any number, $c \neq 0$, and $n$ is odd, $n \geq 3$, then $c$ has exactly one $n$th root.

*Proof (a)*

Let $n = 2k$, $k \geq 1$.

By Theorem 1, $c$ has one real root, call it $r_1$.

We note that if $r_1$ satisfies the equation $x^{2k} - c = 0$ then so does $-r_1$, since

$$\begin{aligned}(-r_1)^2k - c &= \left[(-r_1)^2\right]^k - c \\ &= \left[r_1^2\right]^k - c \\ &= r_1^{2k} - c \\ &= 0\end{aligned}$$

Now, suppose that $r_3$ is any $n$th root of the number $c$. We will show that either $r_3 = r_1$ or $r_3 = r_2$.

We note that $r_1^{2k} - c = 0$ and $r_3^{2k} - c = 0$ imply that $r_1^{2k} - r_3^{2k} = 0$. Using our factoring formula (1) in a convenient form and substituting $r_1^2$ for $x$ and $r_3^2$ for $y$, we obtain

$$\begin{aligned}r_1^{2k} - r_3^{2k} &= \left(r_1^2\right)^k - \left(r_3^2\right)^k \\ &= \left(r_1^2 - r_3^2\right)\sum_{j=1}^{j=k}\left(r_1^2\right)^{k-j}.\left(r_3^2\right)^{j-1} \\ &= 0\end{aligned} \tag{4}$$

Since

$$\sum_{j=1}^{j=k}\left(r_1^2\right)^{k-j}.\left(r_3^2\right)^{j-1} > 0$$

we must have $r_1^2 - r_3^2 = (r_1 - r_3)(r_1 + r_3) = 0$, which implies that either $r_3 = r_1$ or $r_3 = -r_1 = r_2$. This shows that $c$ has no roots distinct from $r_1$ and $r_2$.

*Proof (b)*

This is obvious, since $x^{2k}$ cannot be equal to a negative number as $x^{2k} = (x^2)^k$, and $x^2 \geq 0$ for all real values of $x$.

*Proof (c)*

Let $n = 2k + 1$, $k \geq 1$. If $c > 0$, then by Theorem 1, $c$ has one real root . If $c < 0$, then $-c > 0$ and hence $-c$ has a real root, by Theorem 1, call this root $r_1$. Then $r_1$ satisfies the equation

$$\mathrm{r}_1^{2k+1} = -c$$

Let $r_2 = -r_1$. Then

$$r_2^{2k+1} = -r_1^{2k+1} = -(-c) = c$$

This shows that $r_2$ is an $n$th root of $c$. We have shown that if $n$ is odd, then any real number $c$ has an $n$th root.

Now we show that for $n = 2k + 1$, no non-zero real number $c$ can have two distinct real $n$th roots. Suppose that $c$ does have two distinct real $n$th roots $r_1$ and $r_2$. We will show that this leads to a contradiction.

Since $r_1^{2k+1} - c = 0 = r_2^{2k+1} - c$, it follows that $r_1^{2k+1} - r_2^{2k+1} = 0$. Using the factoring formula, we obtain

$$r_1^{2k+1} - r_2^{2k+1} = \left(r_1 - r_2\right) \sum_{j=1}^{j=2k+1} r_1^{2k+1-j} r_2^{j-1} \qquad (5)$$

We note that when $n$ is odd and $c$ is positive, then $c$ cannot have a negative root since an odd power of a negative number cannot equal a positive number. Similarly, if $c$ is negative, then it cannot have a positive $n$th root. Therefore, either both $r_1$ and $r_2$ are positive or they are both negative. If they are both positive, then certainly

$$\sum_{j=1}^{j=2k+1} r_1^{2k+1-j} r_2^{j-1} > 0$$

If they are both negative, then we can write $r_1 = (-1)|r_1|$ and $r_2 = (-1)|r_2|$, yielding

$$r_1^{2k+1-j}\, r_2^{j-1} = (-1)^{2k+1-j} \mid r_1 \mid^{2k+1-j} . (-1)^{j-1} \mid r_2 \mid^{j-1} = (-1)^{2k} \mid r_1 \mid^{2k+1-j} \mid r_2 \mid^{j-1} > 0$$

Therefore, whether the roots are positive or negative, the sum

$$\sum_{j=1}^{j=k} \left(r_1^2\right)^{k-j} . \left(r_3^2\right)^{j-1} > 0$$

Hence it follows from (5) that we must have $r_1 - r_2 = 0$, contradiction to the assumption that they are distinct.

## Appendix

An alternate proof of the main existence theorem.
Let us first recall some properties of real numbers and definitions.

**Definition**
A number $b$ is called an *upper bound* of a non-empty set $S$ if $s \leq b$ for all $s$ in $S$. An upper bound $b$ of $S$ is called the *least upper bound* (lub) or *supremum* (sup) of $S$ if for any upper bound $c$ of $S$, $b \leq c$.

**Completeness property of real numbers**
If a non-empty set $S$ of real numbers is bounded above, then it has a least upper bound.

**Archimedean property**

For any real number $x$ there exists a natural number $n$ such that $x < n$.

**Trichotomy (of inequality)**

For any two real numbers $x$ and $y$, one of the following holds:

$$x > y,\ x < y,\ x = y$$

Here we only prove the key result stated in Theorem 1 without using the intermediate value theorem, and then the complete analysis will follow the same way as before. But first, we demonstrate the method by giving a couple illustrative simple examples.

**Example 3**

The number 5 has a real cube root.

Let $S = \{\text{real numbers } s{:}s^3 < 5\}$. Then $S \neq \emptyset$, since, for example, 1 is in $S$. $S$ is also bounded above, for example, by the number 2. For, $s > 2$ would imply that $s^3 > 8 > 5$, which would show that $s \notin S$. Therefore, by the completeness property, Sup $S$ exists. Let $t = \sup S$. Then, by the trichotomy property, $t^3$ satisfies one of the following three mutually exclusive inequalities:

(I) $t^3 < 5$

(II) $t^3 > 5$

(III) $t^3 = 5$

We now show that the first two options do not hold.

**Case (I)**

We will show that (I) implies the existence of a natural number $k$ such that $t + \frac{1}{k}$ is in $S$, contradiction to $t$ being an upper bound of $S$. To this end, using the binomial theorem, we write

$$\left(t+\frac{1}{k}\right)^3 = t^3 + 3t^2\frac{1}{k} + 3t\left(\frac{1}{k}\right)^2 + \left(\frac{1}{k}\right)^3$$

We note that $t^3 + 3t^2\frac{1}{k} + 3t\left(\frac{1}{k}\right)^2 + \left(\frac{1}{k}\right)^3 < t^3 + 3t^2\frac{1}{k} + 3t\left(\frac{1}{k}\right) + \frac{1}{k}$

Therefore, in order to show that $\left(t+\frac{1}{k}\right)^3$, it suffices to show the existence of a natural number $k$ satisfying the inequality

$$t^3 + 3t^2\frac{1}{k} + 3t\frac{1}{k} + \frac{1}{k} < 5$$

which is equivalent to the inequality

$$\frac{1}{k}\left(3t^2 + 3t + 1\right) < 5 - t^3$$

or $$k > \frac{3t^2 + 3t + 1}{5 - t^3}$$

The existence of such a number $k$ follows from Archiomedean property. Notice that $5 - t^3 > 0$ by assumption (I). This shows that $t + \frac{1}{k}$ is in $S$, a contradiction.

**Case (II)**

In this case, we wish to show the existence of a natural number $k$ such that $\left(t - \frac{1}{k}\right)^3 > 5$ This will show that $t - \frac{1}{k}$ is an upper bound of $S$ because $s > t - \frac{1}{k}$ would imply $s^3 > \left(t - \frac{1}{k}\right)^3 > 5$ and hence $s$ is not in $S$. We note that

$$\left(t - \frac{1}{k}\right)^3 = t^3 - 3t^2\frac{1}{k} + 3t\left(\frac{1}{k}\right)^2 - \left(\frac{1}{k}\right)^3 > t^3 - 3t^2\frac{1}{k} - \frac{1}{k}$$

Thus it suffices to show the existence of a natural number $k$ satisfying the inequality

$$t^3 - 3t^2\frac{1}{k} - \frac{1}{k} > 5$$

which is equivalent to $$t^3 - 5 > \frac{1}{k}\left(3t^2 + 1\right)$$

or $$k > \frac{3t^2 + 1}{t^3 - 5}$$

Notice that $t^3 - 5$ is positive by (II). It is now clear that $k$ satisfying the last inequality exists by Archimedean property. This rules out the second possibility. Therefore, we must have $t^3 = 5$.

Although the proof of the general theorem uses arguments similar to those given in the examples, there are still some minor technicalities that one has to adjust. So, for the sake of completeness, we wish to record the general result below.

**Theorem 1**

Every positive real number $c$ has at least one positive $n$th root, where $n$ is a natural number, $n \geq 2$.

**Proof**

Let $S = \{\text{real numbers } s{:}s^n < c\}$. Then $S \neq \varnothing$; since, for example, 0 is in $S$. $S$ is also bounded above; for example, by the number $c + 1$. For, $s > c + 1$ would imply that $s^n > (c + 1)^n > c + 1$, by Lemma 1, and hence $s^n > c$, which would show that $s \notin S$. Therefore, by the completeness property, sup $S$ exists. Let $t = \sup S$. Then, by the trichotomy property, $t^n$ satisfies one of the following three inequalities:

(I) $t^n < c$

(II) $t^n > c$

(III) $t^n = c$.

As in the above example, we now show that the first two options do not hold.

**Case (I)**

First, we notice that $S$ contains some positive numbers such as $\frac{c}{c+1}$. Therefore $t > 0$. We now show that $t^n < c$ implies that there exists a natural number $k$ such that $t + \frac{1}{k} \notin S$, contradicting that $t$ is an upper bound of $S$. By the binomial theorem,

$$\left(t+\frac{1}{k}\right)^n = \sum_{j=0}^{j=n} \frac{n!}{(n-j)!\,j!} t^{(n-j)} \left(\frac{1}{k}\right)^j$$

$$= t^n + \sum_{j=1}^{j=n} \frac{n!}{(n-j)!\,j!} t^{(n-j)} \left(\frac{1}{k}\right)^j$$

For any natural number $j$, $\left(\frac{1}{k}\right)^j < \frac{1}{k}$ if $j > 1$. Therefore,

$$\left(t+\frac{1}{k}\right)^n < t^n + \frac{1}{k}.\sum_{j=1}^{j=n} \frac{n!}{(n-j)!\,j!} t^{n-j}$$

For simplicity, let
$$B = \sum_{j=1}^{j=n} \frac{n!}{(n-j)!\,j!} t^{n-j}$$

If we can show that a natural number $k$ exists such that $t^n + \frac{1}{k} B < c$, then we will have shown that $t + \frac{1}{k}$ is in $S$. But $t^n + \frac{1}{k} B < c$ if

$$k > \frac{B}{c - t^n}$$

By assumption (I), $c - t^n > 0$. Thus by Archimedean property, there exists a number $k$ such that

$$k > \frac{B}{c - t^n}$$

This shows that (I) is false.

**Case (II)**

In order to reach a contradiction, we will show the existence of a natural number $k$ such that $\left(t-\frac{1}{k}\right)^n > c$. Here we take $k$ large enough so that $t - \frac{1}{k} > 0$. We note that $s > t - \frac{1}{k}$ would imply that $s^n > \left(t-\frac{1}{k}\right)^n > c$; and therefore $s$ cannot belong to $S$. Thus, $\left(t-\frac{1}{k}\right)^n > c$ would imply that $t - \frac{1}{k}$ is an upper bound of $S$, contradicting the fact that $t$ is the least upper bound of $S$. Now, we note that

$$\left(t-\frac{1}{k}\right)^n = t^n + \sum_{j=1}^{j=n} \frac{n!}{(n-j)!\,j!} t^{n-j} \left(\frac{-1}{k}\right)^j > t^n + \frac{-1}{k} \sum_{j=1}^{j=n} \frac{n!}{(n-j)!\,j!}\ t^{n-j}$$

because $\left(\frac{-1}{k}\right)^j > \frac{-1}{k}$ for $j > 1$. Therefore, it suffices to show the existence of a natural number $k$ such that

$$t^n - \frac{1}{k} \sum_{j=1}^{j=n} \frac{n!}{(n-j)!\,j!}\ t^{n-j} > c$$

For convenience of notation, let

$$B = \sum_{j=1}^{j=n} \frac{n!}{(n-j)!\,j!}\ t^{n-j}$$

Our problem then is reduced to showing the existence of a natural number $k$ such that

$$t^n - \frac{1}{k}B > c$$

or equivalently

$$k > \frac{B}{t^n - c}$$

which follows from Archimedean property. Note that $t^n - c > 0$ by assumption. This completes the proof that option (II) is impossible. Hence $t^n = c$.

## References


Bartle, R. G. & Sherbert, D. R. (2011). *Introduction to real analysis.* New York: John Wiley & Sons.